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# A-schemes and their applications

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# $\mathcal{A}$ -SCHEMES AND THEIR APPLICATIONS

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## 1 Motivation (I)

Let  $(\mathbf{Top})$  (resp.  $(\mathbf{CHaus})$ ) be the category of topological spaces (resp. compact Hausdorff spaces).

### Stone-Čech compactification

$$\beta : (\mathbf{Top}) \rightleftarrows (\mathbf{CHaus}) : U$$

The compact Hausdorff space  $\beta X$  is called the Stone-Čech compactification of the given space  $X$ .

To construct the left adjoint, we need the following lemma:

### Tychonoff's Theorem

$(\mathbf{CHaus})$  is small complete.

We may hope that similar things hold for schemes: schemes are extended concepts of commutative rings. However,

The category  $(\mathbf{Ring})$  of rings is (small) complete and co-complete.

$\Updownarrow$ (compare)

The category  $(\mathbf{Sch})$  of schemes is NOT complete NOR co-complete.

This becomes an obstruction, when we try to construct a universal compactification of schemes using the above method.

**Why is the category of schemes not complete?**

## 2 Motivation (II)

There are several attempts to construct 'scheme-like topological objects' from other algebraic type, such as:

- (1) Monoids ("schemes over  $\mathbb{F}_1$ ")
- (2) Semirings (Tropical geometry).

**What are the common aspects between them?**

## 3 Results

### Properties of $\mathcal{A}$ -schemes (I)

Let  $(\mathbf{CohSch})$  (resp.  $(\mathbf{CohLRS})$ ,  $(\mathcal{A}\text{-}\mathbf{Sch})$ ) be the category of coherent schemes (resp. coherent locally ringed spaces,  $\mathcal{A}$ -schemes).

- (1) Then, we have an adjunction

$$\mathbf{Spec} : (\mathbf{Ring}) \rightleftarrows (\mathcal{A}\text{-}\mathbf{Sch})^{\mathrm{op}} : \Gamma$$

- (2) We have a fully faithful functors

$$(\mathbf{CohSch}) \xrightarrow{j} (\mathcal{A}\text{-}\mathbf{Sch}) \rightarrow (\mathbf{CohLRS}),$$

and  $j$  preserves fibre products and finite quasi-compact open patchings.

### Properties of $\mathcal{A}$ -schemes (II)

- (1) The category  $(\mathcal{A}\text{-}\mathbf{Sch})$  is **small complete and co-complete**.
- (2) We have the notions of **separatedness and properness** (we exclude the 'of finite type' condition). Valuable criteria exist.
- (3) Fix a base  $\mathcal{A}$ -scheme  $S$ , and  $(\mathbf{p}\text{-}\mathcal{A}\text{-}\mathbf{Sch}/S)$  be the category of proper  $\mathcal{A}$ -schemes over  $S$ . Then, we have the adjunction

$$\mathbf{ZR}_S : (\mathcal{A}\text{-}\mathbf{Sch}/S) \rightleftarrows (\mathbf{p}\text{-}\mathcal{A}\text{-}\mathbf{Sch}/S) : U$$

The  $\mathcal{A}$ -scheme  $\mathbf{ZR}_S(X)$  is the **Zariski-Riemann space** associated to  $X$ .

As a corollary, we obtain the well known application for usual schemes:

### Cor (Nagata embedding)

Let  $X \rightarrow S$  be a separated, of finite type morphism of coherent schemes. Then, there exists an open immersion  $X \rightarrow \overline{X}$  of  $S$ -schemes, where  $\overline{X}$  is proper, of finite type over  $S$ .

Unfortunately, we don't have enough space to show here the precise definition of  $\mathcal{A}$ -schemes. The key point is to

**Abandon the feature of schemes, that it is locally a spectrum of a ring.**

## References

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- [2] Takagi, S.: *Nagata embeddings and  $\mathcal{A}$ -schemes*, preprint, arXiv: mathAG/1107.3414